3 (a) À l'aide d'une intégration par parties, montrer que : $\forall n \in \mathbb{N},\ I_{n+1} = \sqrt{2} - (2n+2)\int_0^1 x^{2n+1}\sqrt{x^2+1}\,dx$.

$$I_{n+1} = \int_0^1 \frac{u^{2n+3}}{\sqrt{x^2+1}}\,dx$$

$$= \int_0^1 u^{2n+2} \cdot \frac{u}{\sqrt{x^2+1}}\,dx$$

on pose

$$U(x) = u^{2n+2} \longrightarrow U'(x) = (2n+2)u^{2n+1}$$

$$V'(x) = \frac{u}{\sqrt{x^2+1}} \longrightarrow V(x) = \sqrt{x^2+1}$$

$$I_{n+1} = \left[u^{2n+2}\cdot\sqrt{x^2+1}\right]_0^1 - (2n+2)\cdot\int_0^1 u^{2n+1}\cdot\sqrt{x^2+1}\,dx$$

$$= \sqrt{2} - (2n+2)\cdot\int_0^1 u^{2n+1}\cdot\sqrt{u^2+1}\,dx$$

$\forall n \in \mathbb{N}^*$

$$I_{n+1} = \sqrt{2} - (2n+2)\cdot\int_0^1 u^{2n+1}\sqrt{x^2+1}\,dx$$